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Carine Nourry, Alain Venditti. Local indeterminacy in two-sector overlapping generations models. 2008. halshs-00282832

**HAL Id: halshs-00282832**

**<https://shs.hal.science/halshs-00282832>**

Preprint submitted on 28 May 2008

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Document de Travail  
n°2008-26

## LOCAL INDETERMINACY IN TWO-SECTOR OVERLAPPING GENERATIONS MODELS

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**May 2008**



# Local indeterminacy in two-sector overlapping generations models\*

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First version: *October 2006*; Revised: *March 2008*

**Abstract:** *In this paper, we consider a two-sector two-periods overlapping generations model with inelastic labor, consumption in both periods of life and homothetic CES preferences. We assume in a first step that the consumption levels are gross substitutes and the consumption good is capital intensive. We prove that when dynamic efficiency holds, the occurrence of sunspot fluctuations requires low enough values for the sectoral elasticities of capital-labor substitution. On the contrary, under dynamic inefficiency, local indeterminacy may be obtained without any restriction on the input substitutability properties. Assuming in a second step that gross substitutability in consumption does not hold, we show that sunspot fluctuations arise under dynamic efficiency without any restriction on the sign of the capital intensity difference across sectors and provided the sectoral elasticities of capital-labor substitution admit intermediary values.*

**Keywords:** *Two-sector OLG model, social production function, dynamic (in)efficiency, gross substitutability in consumption, local indeterminacy, sunspot fluctuations*

*Journal of Economic Literature* Classification Numbers: C62, E32, O41.

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\*We would like to thank K. Borissov, J.P. Drugeon, R. Farmer and C. Ghiglino for useful comments and suggestions. This paper also benefited from a presentation at the 4th International Conference on “Economic Theory, Indeterminacy and Non-linear Dynamics in Intertemporal Equilibrium Models”, Tokyo, February 3-4, 2007, and at the International Workshop on “Market Imperfections and Macroeconomic Instability”, University of Lille 1, March 2008.

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# 1 Introduction

The existence of multiple equilibrium paths and sunspot fluctuations are well established facts within OLG models.<sup>1</sup> In particular, in the two-sector formulations with production, Galor [8], Reichlin [11] and Venditti [14] have shown that local indeterminacy may arise under gross substitution in consumption but it requires that the consumption good sector is more capital intensive than the investment good sector.<sup>2</sup>

Reichlin [12] is not directly concerned with the existence of local indeterminacy but shows the possibility of periodic, quasi-periodic and chaotic dynamics in a two-sector model with Leontief technologies. Moreover, the consideration of complementary factors prevents from deriving a general picture of the stability properties of equilibrium paths. Galor [8] and Venditti [14] consider on the contrary general formulations with non-zero factors substitution in production and focuses on the existence of periodic cycles and sunspot fluctuations. However, as they provide implicit sufficient conditions on the main elasticities characterizing the saving function and the technological side, it is very difficult to know whether or not there exists a non-empty set of economies with locally indeterminate equilibria.

This last point is crucial. Indeed, in a recent companion paper (Drugeon *et al.* [7]), we have proved that the existence of sunspot fluctuations is difficult to obtain when gross substitution in consumption holds and the equilibrium path is dynamically efficient. More precisely, we have shown on the one hand, that local indeterminacy is ruled out under a slightly stronger condition than the one ensuring dynamic efficiency, and on the other hand, that saddle-point stability is obtained if the productive factors are sufficiently substitutable. Our main objective in the current paper is thus first to provide clear-cut conditions on the saving function and the elasticities of capital-labor substitution for the occurrence of sunspot fluctuations, and second to articulate the dynamic efficiency and local indeterminacy analysis.

Following Venditti [14], we consider a formulation of the two-sector OLG model based on a social production function which characterizes the factor-price frontier associated with interior temporary equilibria.<sup>3</sup> In order to keep a tractable analysis, we introduce a crucial simplifying assumption: we consider a CES life-cycle utility function which is linearly homogeneous with respect to young and old consumptions. The saving function is thus

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<sup>1</sup>See Woodford [15].

<sup>2</sup>See also Calvo [5].

<sup>3</sup>Such a formulation is the standard way to analyze multisector optimal growth models (see Burmeister *et al.* [4]).

linear with respect to the wage rate, and the propensity to consume, or equivalently the share of first period consumption over the wage income, only depends on the gross rate of return on financial assets. Building on this property, we follow the same procedure as in Drugeon *et al.* [7], but the consideration of a CES utility function allows to generalize their conclusions. Indeed, for any non-unitary value of the elasticity of intertemporal substitution in consumption, we use a scaling parameter to prove the existence of a normalized steady state which remains invariant while the elasticity of intertemporal substitution in consumption is varied, and we show that this normalized steady state is lower than the Golden-Rule capital stock if and only if the share of first period consumption over the wage income is large enough.<sup>4</sup> We finally prove that under this condition, any competitive equilibrium converging to the normalized steady state is dynamically efficient.

In a first step, the local stability analysis of the normalized steady state is performed under the standard assumption of gross substitution in consumption,<sup>5</sup> and with a capital intensive consumption good sector.<sup>6</sup> When dynamic efficiency holds for the normalized steady state, we show that the occurrence of sunspot fluctuations is fundamentally based on intermediary values for the elasticity of intertemporal substitution in consumption and low enough sectoral elasticities of capital-labor substitution. On the contrary, under dynamic inefficiency, we prove that local indeterminacy may be obtained without any restriction on the input substitutability properties, provided the elasticity of intertemporal substitution in consumption is large enough.

In a second step, we eliminate the gross substitutability assumption and we examine the local determinacy properties of the normalized steady state when the elasticity of intertemporal substitution is lower than unity. As already shown by Galor [8] and Venditti [14], the existence of sunspot fluctuations does not require any restriction on the sign of the capital intensity difference across sectors. We then prove that there exists a large set of values for the elasticity of intertemporal substitution in consumption for which local indeterminacy arises under dynamic efficiency provided the sectoral elasticities of capital-labor substitution admit intermediary values.

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<sup>4</sup>In Drugeon *et al.* [7], the existence of a normalized steady state and the dynamic efficiency property are established under the gross substitutability assumption.

<sup>5</sup>The elasticity of intertemporal substitution in consumption is then larger than unity and the propensity to consume is a decreasing function of the gross rate of return.

<sup>6</sup>Under gross substitutability, this is a necessary condition for the occurrence of local indeterminacy (see Galor [8], Venditti [14]).

This paper is organized as follows: The next section sets up the basic model and presents the social production function formulation. In Section 3 we prove the existence of a normalized steady state and we give conditions for dynamic efficiency of the intertemporal competitive equilibrium. Section 4 provides the characteristic polynomial and presents the geometrical method used for the local stability analysis. In Section 5 we present our main results on the joint analysis of dynamic efficiency and local determinacy when gross substitutability holds. Section 6 provides conclusions when gross substitutability is eliminated and the elasticity of intertemporal substitution in consumption is lower than unity. Section 7 presents comparisons with related papers from the literature. All the proofs are gathered in final appendix.

## 2 The model

### 2.1 Production

There are two produced goods, one consumption good  $y_0$  and one capital good  $y$ . The consumption good cannot be used as capital so it is entirely consumed, and the capital good cannot be consumed. There are two inputs, capital and labor. We assume complete depreciation of capital within one period and that labor is inelastically supplied. Each good is produced with a standard constant returns to scale technology:

$$y_0 = f^0(k_0, l_0) \quad (1)$$

$$y = f^1(k_1, l_1) \quad (2)$$

with

$$k_0 + k_1 \leq k, \quad l_0 + l_1 \leq \ell \quad (3)$$

$k$  being the total stock of capital and  $\ell$  the total amount of labor.

**Assumption 1.** *Each production function  $f^i : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ ,  $i = 0, 1$ , is  $C^2$ , increasing in each argument, concave, homogeneous of degree one and such that for any  $x > 0$ ,  $f_1^i(0, x) = f_2^i(x, 0) = +\infty$ ,  $f_1^i(+\infty, x) = f_2^i(x, +\infty) = 0$ .*

Notice that by definition we have  $y \leq f^1(k, \ell)$ . Assumption 1 then implies that there exists  $\bar{k} > 0$  solution of  $k - f^1(k, \ell) = 0$  such that  $f^1(k, \ell) > k$  when  $k < \bar{k}$ , while  $f^1(k, \ell) < k$  when  $k > \bar{k}$ . It follows that it is not possible to maintain stocks over  $\bar{k}$ . The set of admissible 3-uples  $(k, y, \ell)$  is thus defined as follows

$$\tilde{\mathcal{K}} = \{(k, y, \ell) \in \mathbb{R}_+^3 | 0 < \ell, 0 \leq k \leq \bar{k}, 0 \leq y \leq f^1(k, \ell)\} \quad (4)$$

There are two representative firms, one for each sector. For any given  $(k, y, \ell)$ , profit maximization in each representative firm is equivalent to solving the following problem of optimal allocation of productive factors between the two sectors:

$$\begin{aligned} T(k, y, \ell) = \max_{k_0, k_1, l_0, l_1} \quad & f^0(k_0, l_0) \\ \text{s.t.} \quad & (2), (3) \text{ and } k_0, k_1, l_0, l_1 \geq 0 \end{aligned} \quad (5)$$

The social production function  $T(k, y, \ell)$  describes the frontier of the production possibility set associated with interior temporary equilibria such that  $(k, y, \ell) \in \tilde{\mathcal{K}}$ , and gives the maximal output of the consumption good. Under Assumption 1, for any  $(k, y, \ell) \in \tilde{\mathcal{K}}$ ,  $T(k, y, \ell)$  is homogeneous of degree one, concave and we assume in the following that it is  $C^2$ . Denoting  $w$  the wage rate,  $r$  the gross rental rate of capital and  $p$  the price of investment good, all in terms of the price of the consumption good, we derive from the envelope theorem that the first derivatives of the social production function give

$$r = T_1(k, y, \ell), \quad p = -T_2(k, y, \ell), \quad w = T_3(k, y, \ell) \quad (6)$$

Buiding on the homogeneity of  $T(k, y, \ell)$ , the share of capital in total income is then given by

$$s(k, y, \ell) = \frac{rk}{T(k, y, \ell) + py} \in (0, 1) \quad (7)$$

## 2.2 Consumption and savings

In each period  $t$ ,  $N_t$  agents are born, and they live for two periods. In their first period of life (when *young*), the agents are endowed with one unit of labor that they supply inelastically to firms. Their income directly results from the real wage. They allocate this income between current consumption and savings which are invested in the firms. In their second period of life (when *old*), they are retired. Their income is given by the return on the savings made at time  $t$ . As they do not care about events occurring after their death, they consume their income entirely. The preferences of a representative agent born at time  $t$  are thus defined over his consumption bundle  $(c_t, \text{ when he is young, and } d_{t+1}, \text{ when he is old})$  and are summarized by the following CES utility function

$$u(c_t, d_{t+1}) = \left[ c_t^{1-1/\gamma} + \delta(d_{t+1}/B)^{1-1/\gamma} \right]^{\gamma/(\gamma-1)}$$

with  $\delta > 0$  a weighting parameter,  $\gamma > 0$  the elasticity of intertemporal substitution in consumption and  $B > 0$  a scaling parameter.

Each agent is assumed to have  $1 + n > 0$  children so that population is increasing at constant rate  $n$ , *i.e.*,  $N_{t+1} = (1+n)N_t$ . Under perfect foresight,

and considering  $w_t$  and  $R_{t+1}$  as given, a young agent maximizes his utility function over his life-cycle as follows:

$$\begin{aligned} \max_{c_t, d_{t+1}, \phi_t} \quad & u(c_t, d_{t+1}) \\ \text{s.t.} \quad & w_t = c_t + \phi_t \\ & R_{t+1}\phi_t = d_{t+1} \end{aligned} \quad (8)$$

Using the homogeneity of  $u(c_t, d_{t+1})$ , the first order conditions state as:

$$\left(\frac{d_{t+1}}{c_t B}\right)^{1/\gamma} \delta^{-1} = R_{t+1}/B \quad (9)$$

$$c_t + \frac{d_{t+1}}{R_{t+1}} = w_t \quad (10)$$

It follows that:

$$d_{t+1}/c_t B = (\delta R_{t+1}/B)^\gamma \quad (11)$$

Combining (10) and (11), we then derive:

$$c_t = \frac{w_t}{1 + \delta^\gamma (R_{t+1}/B)^{\gamma-1}} \equiv \alpha(R_{t+1}/B)w_t \quad (12)$$

with  $\alpha(R/B) \in (0, 1)$  the propensity to consume of the young, or equivalently the share of first period consumption over the wage income. We also derive the optimal saving as:

$$\phi_t = (1 - \alpha(R_{t+1}/B))w_t \quad (13)$$

Depending on whether the elasticity of intertemporal substitution in consumption  $\gamma$  is larger or lower than 1, the saving function (13) is monotone increasing or decreasing with respect to the gross rate of return  $R$ .

### 2.3 Perfect-foresight competitive equilibrium

Total labor is given by the number  $N_t$  of young households, *i.e.*,  $\ell_t = N_t$ , and is thus increasing at rate  $n$ , *i.e.*,  $\ell_{t+1} = (1 + n)\ell_t$ . We then define a perfect-foresight competitive equilibrium:

**Definition 1.** A sequence  $\{k_t^0, k_t^1, l_t^0, l_t^1, k_t, y_t, \ell_t, c_t, d_t, r_t, w_t, p_t\}_{t=0}^\infty$ , with  $(k_0, \ell_0) = (\hat{k}_0, \hat{\ell}_0)$  given, is a perfect-foresight competitive equilibrium if:

i)  $\{k_t^0, k_t^1, l_t^0, l_t^1\}$  solves (5) given  $(k_t, y_t, \ell_t) \in \tilde{\mathcal{K}}$ ;

ii)  $c_t = \alpha(R_{t+1}/B)w_t$ ;

iii)  $\ell_t(1 - \alpha(R_{t+1}/B))w_t = p_t y_t$ ;

iv)  $y_t = k_{t+1}$ ;

v)  $\ell_{t+1} = (1 + n)\ell_t$ ;

vi)  $\ell_t[c_t + d_t/(1 + n)] = T(k_t, y_t, \ell_t)$ ;



- vii)  $(r_t, w_t, p_t)$  is given by (6);  
 viii)  $R_{t+1} = r_{t+1}/p_t$ .<sup>7</sup>

Let us denote  $\kappa_t = k_t/\ell_t$  the capital-labor ratio at time  $t \geq 0$  and  $\bar{\kappa}$ , solution of  $\kappa - f^1(\kappa, 1) = 0$ , the maximal admissible value of  $\kappa$ . The set of admissible paths given by (4) can then be redefined as follows

$$\mathcal{K} = \{(\kappa_t, \kappa_{t+1}) \in \mathbb{R}_+^2 | 0 \leq \kappa_t \leq \bar{\kappa}, 0 \leq \kappa_{t+1} \leq f^1(\kappa_t, 1)/(1+n)\} \quad (14)$$

Since  $T(k, y, \ell)$  is linearly homogeneous, we derive from Definition 1 that a perfect-foresight competitive equilibrium satisfies the following difference equation:

$$(1+n)\kappa_{t+1} + \frac{T_3(\kappa_t, (1+n)\kappa_{t+1}, 1)}{T_2(\kappa_t, (1+n)\kappa_{t+1}, 1)} \left[ 1 - \alpha \left( -\frac{T_1(\kappa_{t+1}, (1+n)\kappa_{t+2}, 1)}{T_2(\kappa_t, (1+n)\kappa_{t+1}, 1)B} \right) \right] = 0 \quad (15)$$

with  $\alpha(R/B)$  given by (12),  $(\kappa_t, \kappa_{t+1}) \in \mathcal{K}$  and  $\kappa_0 = \hat{\kappa}_0 = \hat{k}_0/\hat{\ell}_0$  given.

### 3 Steady state and dynamic efficiency

#### 3.1 A normalized steady state

A steady state is defined as  $\kappa_t = \kappa^*$ ,  $p_t = p^* = -T_2(\kappa^*, (1+n)\kappa^*, 1)$ ,  $r_t = r^* = T_1(\kappa^*, (1+n)\kappa^*, 1)$ ,  $w_t = w^* = T_3(\kappa^*, (1+n)\kappa^*, 1)$  and  $R^* = r^*/p^*$  for all  $t$  with  $\kappa^*$  solution of the following equation

$$(1+n)\kappa + \frac{T_3(\kappa, (1+n)\kappa, 1)}{T_2(\kappa, (1+n)\kappa, 1)} \left[ 1 - \alpha \left( -\frac{T_1(\kappa, (1+n)\kappa, 1)}{T_2(\kappa, (1+n)\kappa, 1)B} \right) \right] = 0 \quad (16)$$

We will consider in the following a family of economies parameterized by the elasticity of intertemporal substitution in consumption  $\gamma$ . We follow the same procedure as in Drugeon *et al.* [7]: building on the homogeneity property of the utility function, we use the scaling parameter  $B$  in order to give conditions for the existence of a normalized steady state  $\kappa^* \in (0, \bar{\kappa})$  which will remain invariant as the elasticity of intertemporal substitution in consumption is varied. However, we need also to ensure that the value of the share of first period consumption over the wage income  $\alpha(R/B)$  when evaluated at the normalized steady state does not depend on  $\gamma$ . This property will be obtained by choosing adequately the value of the weighting parameter  $\delta$ . Proceeding that way, for a given set of parameters characterizing the

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<sup>7</sup>Starting from the equality *vi*) in Definition 1 and using the budget constraints of the representative agent with the homogeneity of  $T(k, y, \ell)$  we get  $\ell_{t+1}(w_{t+1} - \phi_{t+1} + R_{t+1}\phi_t) = r_{t+1}k_{t+1} - p_{t+1}y_{t+1} + w_{t+1}\ell_{t+1}$ . The result is obtained after obvious simplifications.

technologies and preferences, we will be able to isolate the role of  $\gamma$  on the local stability properties of competitive equilibria.

We will consider in the following either  $\gamma > 1$  in the case of gross substitutability between the consumption levels  $c_t$  and  $d_{t+1}$ , or  $\gamma \in (0, 1)$  if we do not assume this property.<sup>8</sup> We then get:

**Proposition 1.** *Under Assumption 1, let  $\gamma \in \mathbb{R}_+^*/\{1\}$ ,  $\kappa^* \in (0, \bar{\kappa})$  and  $\delta = -T_2(\kappa^*, (1+n)\kappa^*, 1)B/T_1(\kappa^*, (1+n)\kappa^*, 1)$ . Then there exists a unique value  $B^* > 0$  as given by*

$$B^* = \frac{(1+n)\kappa^*T_1(\kappa^*, (1+n)\kappa^*, 1)}{T_3(\kappa^*, (1+n)\kappa^*, 1) + (1+n)\kappa^*T_2(\kappa^*, (1+n)\kappa^*, 1)} \quad (17)$$

such that  $\kappa^*$  is a steady state if and only if  $B = B^*$ .

*Proof:* See Appendix 8.1. □

In the rest of the paper we will assume that  $B = B^*$  and  $\delta = -T_2(\kappa^*, (1+n)\kappa^*, 1)B^*/T_1(\kappa^*, (1+n)\kappa^*, 1) \equiv \delta^*$  in order to guarantee the existence of one normalized steady state (NSS in the sequel).

### 3.2 Dynamic (in)efficiency

Let us evaluate all the shares and elasticities previously defined at the NSS. From (7), (12) and considering  $B = B^*$  and  $\delta = \delta^*$  as given by Proposition 1, let  $s = s(\kappa^*, \kappa^*, 1)$  and  $\alpha = \alpha(-T_1(\kappa^*, (1+n)\kappa^*, 1)/T_2(\kappa^*, (1+n)\kappa^*, 1)B^*)$ .

Drugeon *et al.* [7] provide an analysis of the dynamic efficiency properties of equilibrium paths based on a NSS characterized by an under-accumulation of capital with respect to the Golden Rule. From Definition 1, (17) and the homogeneity of  $T(k, y, \ell)$ , considering that  $\kappa^*T_2/T_3 = (T_2/T_1)(\kappa^*T_1/T_3) = -s/R(1-s)$ , we derive the stationary gross rate of return along the NSS:

$$R^* = \frac{(1+n)s}{(1-\alpha)(1-s)} \quad (18)$$

Under-accumulation of capital is obtained if and only if  $R^* > 1+n$ . Considering that with a CES utility function, Proposition 1 ensures the existence of a NSS for any  $\gamma \in \mathbb{R}_+^*/\{1\}$ , the proof of Proposition 2 in Drugeon *et al.* [7], derived under  $\gamma > 1$ , also covers the case  $\gamma \in (0, 1)$ . We then get:

**Proposition 2.** *Under Assumption 1, let  $\gamma \in \mathbb{R}_+^*/\{1\}$  and  $\underline{\alpha} = 1-s/(1-s)$ . Then:*

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<sup>8</sup>The Cobb-Douglas formulation with  $\gamma = 1$  will not be considered since the saving function then does not depend on the gross rate of return  $R$  and  $\alpha(z) = \alpha$  is constant.

- i) the NSS is characterized by an under-accumulation of capital if and only if  $\alpha \geq \underline{\alpha}$ ;*
- ii) an intertemporal competitive equilibrium converging towards the NSS is dynamically efficient if  $\alpha \in (\underline{\alpha}, 1)$  and dynamically inefficient if  $\alpha \in (0, \underline{\alpha})$ .*

Notice from the definition of the bound  $\underline{\alpha}$  that if the labor income is relatively lower than the capital income, i.e.,  $s \geq 1/2$ , then a young agent does not have enough wage resources to provide a large amount of savings so that an under-accumulation of capital is obtained without additional restriction. On the contrary, if the labor income is relatively larger than the capital income, i.e.,  $s < 1/2$ , then a young agent receives enough wage resources to be able to provide a large amount of savings. In this case, over-accumulation of capital can be avoided provided his share of first period consumption over the wage income is large enough.

In the rest of the paper we will restrict the share of capital in total income in order to get a positive value for the bound  $\underline{\alpha} = 1 - s/(1 - s)$ :

**Assumption 2.**  $s \in (0, 1/2)$ .

## 4 Local properties of the normalized steady state

### 4.1 Characteristic polynomial

In order to derive a tractable formulation for the characteristic polynomial, we introduce the relative capital intensity difference across sectors

$$b \equiv \frac{l^1}{y} \left( \frac{k^1}{l^1} - \frac{k^0}{l^0} \right) \quad (19)$$

and the elasticity of the rental rate of capital

$$\varepsilon_{rk} = -T_{11}(\kappa^*, (1+n)\kappa^*, 1)\kappa^*/T_1(\kappa^*, (1+n)\kappa^*, 1) \quad (20)$$

evaluated at the NSS. Assuming that  $b \neq 0$ , let us linearize the difference equation (15) around the NSS:

**Lemma 1.** *Under Assumption 1, the characteristic polynomial is*

$$\mathcal{P}(\lambda) = \lambda^2 - \lambda\mathcal{T} + \mathcal{D} \quad (21)$$

with

$$\mathcal{D} = \frac{s[(1+n)b\alpha(\gamma-1)+1-\alpha+\alpha(1+n)b]}{(1+n)b(1-\alpha)(1-s)\alpha(\gamma-1)}, \quad \mathcal{T} = \frac{1}{(1+n)b\varepsilon_{rk}\alpha(\gamma-1)} + \frac{1+\mathcal{D}(1+n)^2b^2}{(1+n)b}$$

*Proof:* See Appendix 8.2. □

## 4.2 Geometrical method

Under Assumption 1, Proposition 1 shows that when the scaling and weighting parameters satisfy  $B = B^*$  and  $\delta = \delta^*$ , the NSS and all the other shares and elasticities characterizing preferences and technologies remain constant as the elasticity of intertemporal substitution  $\gamma$  is made to vary. As in Drugeon *et al.* [7], we will then study the variations of the trace  $\mathcal{T}(\gamma)$  and the determinant  $\mathcal{D}(\gamma)$  in the  $(\mathcal{T}, \mathcal{D})$  plane as  $\gamma$  varies continuously within  $(1, +\infty)$ . This methodology, initially presented in Grandmont *et al.* [9], allows to easily characterize the local stability of the NSS, as well as the occurrence of local bifurcations. Indeed, from Proposition 1, solving  $\mathcal{T}$  and  $\mathcal{D}$  with respect to  $\alpha(\gamma - 1)$  yields to the following linear relationship  $\Delta(\mathcal{T})$ :

$$\mathcal{D} = \Delta(\mathcal{T}) = \mathcal{S}\mathcal{T} - \frac{\varepsilon_{rk}s[1-\alpha+\alpha(1+n)b]-s(1+n)b}{(1+n)b[(1-\alpha)(1-s)+\varepsilon_{rk}s(1+n)b(1-\alpha+\alpha(1+n)b)]} \quad (22)$$

where the slope  $\mathcal{S}$  of  $\Delta(\mathcal{T})$  is

$$\mathcal{S} = \frac{\varepsilon_{rk}s[1-\alpha+\alpha(1+n)b]}{(1-\alpha)(1-s)+\varepsilon_{rk}s(1+n)b[1-\alpha+\alpha(1+n)b]}. \quad (23)$$

As  $\gamma$  spans the interval  $(1, +\infty)$ ,  $\mathcal{T}(\gamma)$  and  $\mathcal{D}(\gamma)$  vary linearly along the line  $\Delta(\mathcal{T})$ . Figure 1 provides an illustration of  $\Delta(\mathcal{T})$ .

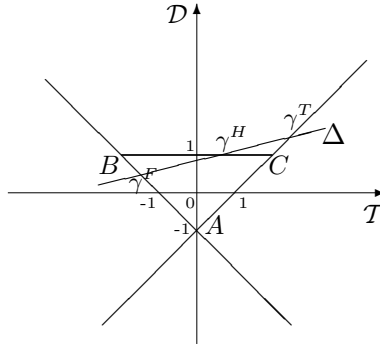


Figure 1: Stability triangle and  $\Delta(\mathcal{T})$  line.

We also introduce three other relevant lines: line  $AC$  ( $\mathcal{D} = \mathcal{T} - 1$ ) along which one characteristic root is equal to 1, line  $AB$  ( $\mathcal{D} = -\mathcal{T} - 1$ ) along which one characteristic root is equal to  $-1$  and segment  $BC$  ( $\mathcal{D} = 1, |\mathcal{T}| < 2$ ) along which the characteristic roots are complex conjugate with modulus equal to 1. These lines divide the space  $(\mathcal{T}, \mathcal{D})$  into three different types of regions according to the number of characteristic roots with modulus less than 1. When  $(\mathcal{T}, \mathcal{D})$  belongs to the interior of triangle  $ABC$ , the NSS is locally indeterminate. Let  $\gamma^F$ ,  $\gamma^T$  and  $\gamma^H$  in  $(1, +\infty)$  be the values of  $\gamma$

at which  $\Delta(\mathcal{T})$  respectively crosses the lines  $AB$ ,  $AC$  and the segment  $BC$ . Then as  $\gamma$  respectively goes through  $\gamma^F$ ,  $\gamma^T$  or  $\gamma^H$ , a flip, transcritical or Hopf bifurcation generically occurs.<sup>9</sup>

Under gross substitutability, *i.e.*,  $\gamma > 1$ , we know since Galor [8] that local indeterminacy necessarily requires a capital intensive consumption good, *i.e.*,  $b < 0$ , but Hopf bifurcation is ruled out since the characteristic roots are real. On the contrary, if gross substitutability does not hold, *i.e.*,  $\gamma \in (0, 1)$ , local indeterminacy may also occur with a capital intensive investment good, *i.e.*,  $b > 0$ , and Hopf bifurcation cannot be ruled out when  $b < 0$ .<sup>10</sup>

In a first step we will focus on the standard case with gross substitutability in which the saving function is increasing with respect to the gross rate of return  $R$ . Since we are interested in the existence of local indeterminacy, we also assume that the consumption good is capital intensive.

**Assumption 3.**  $\gamma > 1$  and  $b < 0$ .

The case  $\gamma \in (0, 1)$ , which is related to the existence of local indeterminacy with  $b > 0$  or the existence of a Hopf bifurcation with  $b < 0$ , will be partially discussed in Section 6.

As  $\gamma \in (1, +\infty)$ , the fundamental properties of  $\Delta(\mathcal{T})$  are characterized from the consideration of its extremities. The starting point of the pair  $(\mathcal{T}(\gamma), \mathcal{D}(\gamma))$  is indeed obtained when  $\gamma = +\infty$ :

$$\begin{aligned} \lim_{\gamma \rightarrow +\infty} \mathcal{D}(\gamma) &= \mathcal{D}_\infty = \frac{s}{(1-\alpha)(1-s)} \\ \lim_{\gamma \rightarrow +\infty} \mathcal{T}(\gamma) &= \mathcal{T}_\infty = \frac{(1-\alpha)(1-s) + (1+n)^2 b^2 s}{(1+n)b(1-\alpha)(1-s)} \end{aligned} \quad (24)$$

while the end point is obtained when  $\gamma$  converges to 1 from above:

$$\begin{aligned} \lim_{\gamma \rightarrow 1^+} \mathcal{D}(\gamma) &= \mathcal{D}_1^+ = \pm\infty \Leftrightarrow b[1 - \alpha + \alpha(1+n)b] \gtrless 0 \\ \lim_{\gamma \rightarrow 1^+} \mathcal{T}(\gamma) &= \mathcal{T}_1^+ = \pm\infty \\ \Leftrightarrow b[(1-\alpha)(1-s) + \varepsilon_{rk}(1+n)bs[1 - \alpha + \alpha(1+n)b]] &\gtrless 0 \end{aligned} \quad (25)$$

---

<sup>9</sup>When  $\gamma$  goes through  $\gamma^T$ , one characteristic root crosses 1. Proposition 1 shows that the existence of the NSS is always ensured as soon as  $B = B^*$  and a saddle-node bifurcation cannot occur. Depending on the number of steady states, the critical value  $\gamma^T$  will be associated with an exchange of stability between the NSS and another (resp. two others) steady state through a transcritical (resp. pitchfork) bifurcation. However, as shown in Ruelle [13], pitchfork bifurcations require a non-generic condition. In order to simplify the exposition we concentrate on the generic case and we associate in the rest of the paper the existence of one eigenvalue going through 1 to a transcritical bifurcation.

<sup>10</sup>See Galor [8] and Venditti [14].

Moreover, we get

$$\mathcal{D}'(\gamma) = -\frac{s[1-\alpha+\alpha(1+n)b]}{(1+n)b(1-\alpha)(1-s)\alpha(\gamma-1)^2} \quad (26)$$

It follows that  $\mathcal{D}'(\gamma) \geq 0$  if and only if  $\mathcal{D}_1^+ = \mp\infty$ .

The next Lemmas provide a precise characterization of  $\Delta(\mathcal{T})$ . A first one gives informations on the starting point  $(\mathcal{T}_\infty, \mathcal{D}_\infty)$  and  $\mathcal{D}'(\gamma)$ :

**Lemma 2.** *Under Assumptions 1-3, for given  $s, \alpha, b$  and  $\varepsilon_{rk}$ , the following results hold:*

- i)  $\mathcal{D}_\infty > 1$  if and only if  $\alpha > \underline{\alpha}$ ;
- ii)  $\mathcal{D}'(\gamma) > 0$  if and only if  $b \in (-(1-\alpha)/(1+n)\alpha, 0)$ ;
- iii)  $\mathcal{T}_\infty < 0$ ;
- iv) When  $\alpha > \underline{\alpha}$ ,  $1 + \mathcal{T}_\infty + \mathcal{D}_\infty < 0$  if and only if  $b \in (-\infty, -1/(1+n)) \cup (-(1-\alpha)(1-s)/(1+n)s, 0)$ ;
- v) When  $\alpha \in (0, \underline{\alpha})$ ,  $1 + \mathcal{T}_\infty + \mathcal{D}_\infty < 0$  if and only if  $b \in (-\infty, -(1-\alpha)(1-s)/(1+n)s) \cup (-1/(1+n), 0)$ .

*Proof:* See Appendix 8.3. □

Lemma 2 exhibits three critical bounds on  $b$  which appear to be crucial for the stability properties of the NSS:  $b_0 = -1/(1+n)$ ,  $b_1 = -(1-\alpha)/(1+n)\alpha$  and  $b_2 = -(1-\alpha)(1-s)/(1+n)s$ . We obtain the following comparisons:

$$\begin{aligned} b_1 > b_0 &\Leftrightarrow \alpha > 1/2, \quad b_2 > b_1 \Leftrightarrow \alpha < s/(1-s), \\ \underline{\alpha} < 1/2 &\Leftrightarrow s > 1/3, \quad s/(1-s) > 1/2 \Leftrightarrow s > 1/3 \end{aligned} \quad (27)$$

Notice from Lemma 2 that  $\alpha > \underline{\alpha}$  implies  $\mathcal{D}_\infty > 1$  and the existence of local indeterminacy requires  $\mathcal{D}'(\gamma) > 0$ , i.e.  $b \in (b_1, 0)$ . A second Lemma then provides additional informations on the intersections of  $\Delta(\mathcal{T})$  with the lines  $AB$  and  $AC$  when  $b \in (b_1, 0)$ :

**Lemma 3.** *Under Assumptions 1-3, let  $\alpha > \underline{\alpha}$  and  $b \in (b_1, 0)$ . There exists  $\underline{\varepsilon}_{rk} > 0$  such that for given  $s, \alpha, b$  and  $\varepsilon_{rk}$ , the following results hold:*

- 1 -  $\Delta(\mathcal{T}) = 1$  implies  $\mathcal{T} < -2$ .
- 2 -  $\Delta(\mathcal{T}) = -1$  implies  $\mathcal{T} < 0$  in the following cases:
  - i)  $\alpha > \max\{\underline{\alpha}, 1/2\}$ ,
  - ii)  $s \in (1/3, 1/2)$ ,  $\alpha \in (\underline{\alpha}, 1/2)$  and  $b \in (b_0, 0)$ ,
  - iii)  $s \in (1/3, 1/2)$ ,  $\alpha \in (\underline{\alpha}, 1/2)$ ,  $b \in (b_1, b_0)$  and  $\varepsilon_{rk} \in (0, \underline{\varepsilon}_{rk})$ .
- 3 -  $\Delta(\mathcal{T}) = -1$  implies  $\mathcal{T} > 0$  if and only if  $s \in (1/3, 1/2)$ ,  $\alpha \in (\underline{\alpha}, 1/2)$ ,  $b \in (b_1, b_0)$  and  $\varepsilon_{rk} > \underline{\varepsilon}_{rk}$ .

*Proof:* See Appendix 8.4.  $\square$

Notice now from Lemma 2 that when  $\alpha < \underline{\alpha}$  we have  $\mathcal{D}_\infty \in (0, 1)$  and the existence of local indeterminacy may be obtained either with  $\mathcal{D}'(\gamma) > 0$ , i.e.  $b \in (b_1, 0)$ , or with  $\mathcal{D}'(\gamma) < 0$ , i.e.  $b \in (-\infty, b_1)$ . A third Lemma then provides additional informations on the intersections of  $\Delta(\mathcal{T})$  with the lines  $AB$ ,  $BC$  and  $AC$ :

**Lemma 4.** *Under Assumptions 1-3, let  $\alpha \in (0, \underline{\alpha})$ . There exists  $\varepsilon_{rk} > 0$  such that for given  $s$ ,  $\alpha$ ,  $b$  and  $\varepsilon_{rk}$ , the following results hold:*

- 1 - *When  $b \in (-\infty, b_1)$ ,  $\Delta(\mathcal{T}) = 1$  implies  $\mathcal{T} < -2$ .*
- 2 - *When  $b \in (b_1, 0)$ ,  $\Delta(\mathcal{T}) = -1$  implies  $\mathcal{T} < 0$  in the following cases:*
  - i)  *$s \in (0, 1/3)$  and  $\alpha \in (1/2, \underline{\alpha})$ ,*
  - ii)  *$\alpha \in (0, \min\{\underline{\alpha}, 1/2\})$  and  $b \in (b_0, 0)$ ,*
  - iii)  *$\alpha \in (0, \min\{\underline{\alpha}, 1/2\})$ ,  $b \in (b_1, b_0)$  and  $\varepsilon_{rk} \in (0, \varepsilon_{rk})$ .*
- 3 -  *$\Delta(\mathcal{T}) = -1$  implies  $\mathcal{T} > 0$  if and only if  $\alpha \in (0, \min\{\underline{\alpha}, 1/2\})$ ,  $b \in (b_1, b_0)$  and  $\varepsilon_{rk} > \varepsilon_{rk}$ .*

*Proof:* See Appendix 8.5.  $\square$

To sum up, in graphical terms, the relevant part of  $\Delta(\mathcal{T})$  is thus a half-line starting in  $(\mathcal{T}_\infty, \mathcal{D}_\infty)$ , with  $\mathcal{T}_\infty < 0$ ,  $\mathcal{D}_\infty > 0$ , and pointing upwards or downwards, to the right or to the left depending on the sign of  $\mathcal{D}'(\gamma)$ .

## 5 Local indeterminacy

### 5.1 Under dynamic efficiency

If  $\alpha > \underline{\alpha}$ ,  $\Delta(\mathcal{T})$  starts within an area in which local determinacy necessarily holds since  $\mathcal{D}_\infty > 1$ . The possible occurrence of local indeterminacy requires therefore that  $\mathcal{D}(\gamma)$  is an increasing function. Let us first introduce as suggested by Lemma 3 a slightly stronger condition on the share  $\alpha$  by assuming that  $\alpha > \max\{\underline{\alpha}, 1/2\}$ . This inequality implies  $b_1 > b_0$ . While Lemma 2 shows that local indeterminacy might occur when  $b \in (b_1, 0)$ , Lemma 3 implies that when  $\alpha > \max\{\underline{\alpha}, 1/2\}$ ,  $\Delta(\mathcal{T})$  can only cross the line  $AB$  when  $\mathcal{D}(\gamma) > 1$  or the line  $AC$  when  $\mathcal{D}(\gamma) < -1$  and local indeterminacy is ruled out.

**Proposition 3.** *Under Assumptions 1-3, if  $\alpha > \max\{\underline{\alpha}, 1/2\}$ , the NSS is locally determinate.*

Consider then Lemma 2. Since  $\alpha > \underline{\alpha}$  implies  $\mathcal{D}_\infty > 1$ , local indeterminacy requires that  $\mathcal{D}'(\gamma) > 0$ , i.e.  $b \in (b_1, 0)$ . But in this case,  $\mathcal{D} = 1$  implies

$\mathcal{T} < -2$ . As a result, the existence of an intersection between the  $\Delta$ -half-line and the interior of triangle  $ABC$  requires first that  $1 + \mathcal{T}_\infty + \mathcal{D}_\infty < 0$ , i.e.  $b \in (-\infty, b_0) \cup (b_2, 0)$ , and second that  $\mathcal{D} = -1$  implies  $\mathcal{T} > 0$ . As shown by Lemma 3, this last property is obtained if and only if  $s \in (1/3, 1/2)$ ,  $\alpha \in (\underline{\alpha}, 1/2)$ ,  $b \in (b_1, b_0)$  and  $\varepsilon_{rk} > \underline{\varepsilon}_{rk}$ . Under these restrictions, the  $\Delta$ -half-line crosses the ordinate axis within the interior of the segment  $(-1, 0)$  as in the following geometrical representation:

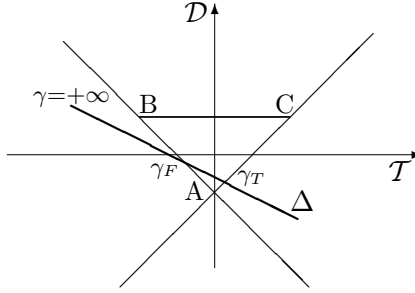


Figure 2: Local indeterminacy with dynamic efficiency.

We then derive

**Proposition 4.** *Under Assumptions 1-3, there exist  $\underline{\varepsilon}_{rk} > 0$ ,  $\gamma_T > 1$  and  $\gamma_F > \gamma_T$  such that the NSS is locally indeterminate if and only if  $s \in (1/3, 1/2)$ ,  $\alpha \in (\underline{\alpha}, 1/2)$ ,  $b \in (b_1, b_0)$ ,  $\varepsilon_{rk} > \underline{\varepsilon}_{rk}$  and  $\gamma \in (\gamma_T, \gamma_F)$ .*

When the competitive equilibrium converging to the NSS is dynamically efficient, local indeterminacy is necessarily associated with the existence of multiple steady states since  $\gamma_T$  is generically a transcritical bifurcation value. It requires also a large enough elasticity of the rental rate of capital. Finally, the occurrence of sunspot fluctuations is fundamentally based on intermediary values for the elasticity of intertemporal substitution in consumption. Notice also that endogenous fluctuations are obtained from a flip bifurcation occurring when  $\gamma$  crosses  $\gamma^F$  from below.

Proposition 4 shows that sunspot fluctuations require a large enough value of  $\varepsilon_{rk}$ . This restriction may be easily interpreted. Denoting  $\sigma_i$  the elasticity of capital-labor substitution in sector  $i = 0, 1$  and using Drugeon [6], we may define an aggregate elasticity of substitution between capital and labor, denoted  $\Sigma$ , which is obtained as a weighted sum of the sectoral elasticities  $\sigma_i$ , and then derive an expression for the elasticity of the interest rate  $\varepsilon_{rk}$ , namely:<sup>11</sup>

<sup>11</sup>The expression (28) is derived from Proposition 2 in Drugeon [6].



$$\Sigma = \frac{y_0 + py}{pyky_0} (pyk_0l_0\sigma_0 + y_0k_1l_1\sigma_1), \quad \varepsilon_{rk} = (l_0/y_0)^2 \frac{w(y_0 + py)}{\Sigma} \quad (28)$$

Therefore, as shown in Propositions 4, local indeterminacy is generally associated with a large elasticity of the rental rate of capital, *i.e.*, low enough sectoral elasticities of capital-labor substitution.

## 5.2 Under dynamic inefficiency

As shown in Proposition 2, dynamic inefficiency is obtained when  $\alpha \in (0, \underline{\alpha})$ . In such a case, Lemma 2 shows that  $\Delta(\mathcal{T})$  starts within an area in which  $\mathcal{D}_\infty \in (0, 1)$ . Different configurations for local indeterminacy may occur depending on whether the starting point  $(\mathcal{T}_\infty, \mathcal{D}_\infty)$  is within the triangle  $ABC$  or not.

Let us start with the configuration in which  $(\mathcal{T}_\infty, \mathcal{D}_\infty)$  belongs to the region where the NSS is saddle-point stable. We know indeed from Lemma 2 that this is the case when  $1 + \mathcal{T}_\infty + \mathcal{D}_\infty < 0$ , *i.e.*,  $b \in (-\infty, b_2) \cup (b_0, 0)$ . Lemma 4 also shows that when  $b \in (-\infty, b_1)$ ,  $\mathcal{D} = 1$  implies  $\mathcal{T} < -2$  so that any  $\Delta$ -half-line pointing upward cannot intersect the triangle  $ABC$ . It follows that local indeterminacy requires  $\mathcal{D}'(\gamma) > 0$ , *i.e.*,  $b \in (b_1, 0)$  as shown by Lemma 2. Moreover we derive from Lemma 4 that  $\Delta(\mathcal{T})$  will cross the triangle  $ABC$  if and only if  $\alpha \in (0, \min\{\underline{\alpha}, 1/2\})$ ,  $b \in (b_1, b_0)$  and  $\varepsilon_{rk} > \underline{\varepsilon}_{rk}$ , *i.e.*, when  $\mathcal{D} = -1$  implies  $\mathcal{T} > 0$ . Notice now from (27) that  $\alpha \in (0, \min\{\underline{\alpha}, 1/2\})$  implies  $b_0 > b_1$ . Therefore, in order to get a compatibility between  $b \in (-\infty, b_2) \cup (b_0, 0)$  and  $b \in (b_1, b_0)$ , we need to have  $b_1 < b_2$ , *i.e.*,  $\alpha < s/(1-s)$  as shown again by (27). To summarize, assuming that  $\alpha \in (0, \min\{\underline{\alpha}, 1/2, s/(1-s)\})$ ,  $b \in (b_1, b_0)$  and  $\varepsilon_{rk} > \underline{\varepsilon}_{rk}$ , we get the following geometrical representation:

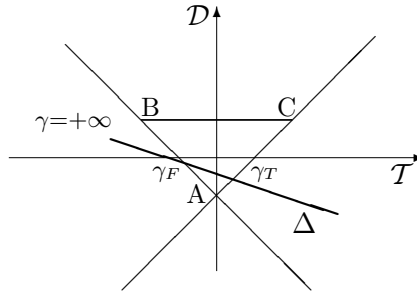


Figure 3:  $\alpha \in (0, \min\{\underline{\alpha}, 1/2, s/(1-s)\})$ ,  $b \in (b_1, b_0)$  and  $\varepsilon_{rk} > \underline{\varepsilon}_{rk}$ .

This configuration is similar to the one obtained under dynamic efficiency. Local indeterminacy is associated with the existence of multiple steady states since  $\gamma_T$  is generically a transcritical bifurcation value, and

endogenous fluctuations are also obtained from a flip bifurcation occurring when  $\gamma$  crosses  $\gamma^F$  from below. Notice finally that the existence of sunspot fluctuations requires intermediary values for the elasticity of intertemporal substitution in consumption.

Let us consider now the configuration in which  $(\mathcal{T}_\infty, \mathcal{D}_\infty)$  belongs to the interior of triangle  $ABC$  so that the NSS is locally indeterminate. We know from Lemma 2 that this is the case when  $1 + \mathcal{T}_\infty + \mathcal{D}_\infty > 0$ , *i.e.*,  $b \in (b_2, b_0)$ . Assume in a first step that the  $\Delta$ -half-line is pointing upward, *i.e.*,  $b \in (-\infty, b_1)$ . Lemma 4 then shows that  $\mathcal{D} = 1$  implies  $\mathcal{T} < -2$ . In order to get a compatibility between  $b \in (b_2, b_0)$  and  $b \in (-\infty, b_1)$ , we need to have  $b_1 > b_2$ , *i.e.*,  $\alpha > s/(1-s)$  as shown by (27). Since  $\alpha \in (0, \underline{\alpha})$ , we have to impose  $s/(1-s) < \underline{\alpha}$ , or equivalently  $s \in (0, 1/3)$ . But this last restriction implies  $\underline{\alpha} > 1/2$  as shown again by (27). Therefore, starting from  $(\mathcal{T}_\infty, \mathcal{D}_\infty)$  within the interior of triangle  $ABC$ , and recalling from (27) that  $b_1 > b_0$  if and only if  $\alpha > 1/2$ ,  $\Delta(\mathcal{T})$  will cross the segment  $AB$  if  $s \in (0, 1/3)$  and one of the following sets of conditions is satisfied:

- i)  $\alpha \in (s/(1-s), 1/2)$  and  $b \in (b_2, b_1)$ ,
- ii)  $\alpha \in (1/2, \underline{\alpha})$  and  $b \in (b_2, b_0)$ .

To summarize, assuming that  $s \in (0, 1/3)$ ,  $\alpha \in (s/(1-s), \underline{\alpha})$  and  $b \in (b_2, \min\{b_1, b_0\})$ , we get the following geometrical representation:

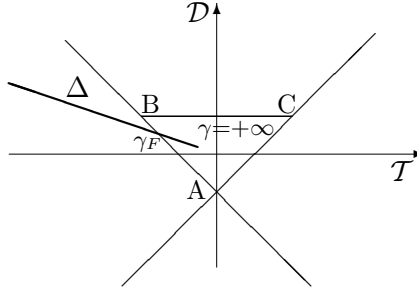


Figure 4:  $s \in (0, 1/3)$ ,  $\alpha \in (s/(1-s), \underline{\alpha})$  and  $b \in (b_2, \min\{b_1, b_0\})$ .

In this configuration, in contradiction with the previous one, local indeterminacy is based on large enough values for the elasticity of intertemporal substitution in consumption, and uniqueness of the steady state is a generic property since there cannot exist any transcritical bifurcation. However, a flip bifurcation and period-two cycles still occur when  $\gamma$  crosses  $\gamma^F$  from above.

Assume finally that  $(\mathcal{T}_\infty, \mathcal{D}_\infty)$  belongs to the interior of triangle  $ABC$ , *i.e.*,  $b \in (b_2, b_0)$ , but the  $\Delta$ -half-line is pointing downward, *i.e.*,  $b \in (b_1, 0)$ . In order to get a compatibility between  $b \in (b_2, b_0)$  and  $b \in (b_1, 0)$ , we

need to have  $b_1 < b_0$ , i.e.,  $\alpha < 1/2$  as shown by (27). Let us then assume  $\alpha \in (0, \min\{\underline{\alpha}, 1/2\})$  and  $b \in (\max\{b_1, b_2\}, b_0)$ . As shown by Lemma 4, depending on whether  $\varepsilon_{rk}$  is lower or larger than  $\underline{\varepsilon}_{rk}$ ,  $\mathcal{D} = -1$  implies  $\mathcal{T} < 0$  or  $\mathcal{T} > 0$  and we get a half-line as given by  $\Delta$  or  $\Delta'$  in the following Figure.

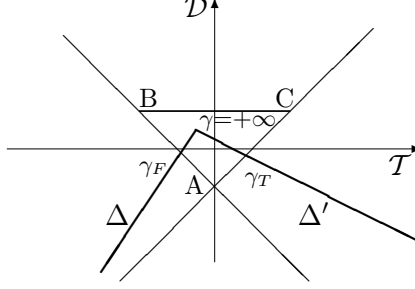


Figure 5:  $\alpha \in (0, \min\{\underline{\alpha}, 1/2\})$  and  $b \in (\max\{b_1, b_2\}, b_0)$ .

In this last configuration, local indeterminacy is again based on large enough values for the elasticity of intertemporal substitution in consumption. The existence of multiple steady states is again a possible outcome since a transcritical bifurcation may occur. Notice finally that depending on whether the elasticity of the rental rate of capital  $\varepsilon_{rk}$  is lower or larger than the bound  $\underline{\varepsilon}_{rk}$ , the NSS becomes saddle-point stable through a flip or a transcritical bifurcation as  $\gamma$  crosses  $\gamma^F$  or  $\gamma^T$  from above.

All these results can be gathered into the following Proposition:

**Proposition 5.** *Under Assumptions 1-3, there exist  $\underline{\varepsilon}_{rk} > 0$ ,  $\underline{\gamma} > 1$  and  $\bar{\gamma} > 1$  such that the NSS is locally indeterminate if and only if one of the following sets of conditions is satisfied:*

- i)  $\alpha \in (0, \min\{\underline{\alpha}, 1/2, s/(1-s)\})$ ,  $b \in (b_1, b_0)$ ,  $\varepsilon_{rk} > \underline{\varepsilon}_{rk}$  and  $\gamma \in (\underline{\gamma}, \bar{\gamma})$ ,
- ii)  $s \in (0, 1/3)$ ,  $\alpha \in (s/(1-s), \underline{\alpha})$ ,  $b \in (b_2, \min\{b_1, b_0\})$  and  $\gamma > \underline{\gamma}$ ,
- iii)  $\alpha \in (0, \min\{\underline{\alpha}, 1/2\})$ ,  $b \in (\max\{b_1, b_2\}, b_0)$  and  $\gamma > \underline{\gamma}$ .

As shown by Figure 3, in case i),  $\underline{\gamma} = \gamma^T$  and  $\bar{\gamma} = \gamma^F$  so that local indeterminacy is fundamentally associated with the existence of multiple steady states and period-two cycles. It requires also intermediary values for the elasticity of intertemporal substitution in consumption. Figure 4 shows on the contrary that in case ii),  $\underline{\gamma} = \gamma^F$  and local indeterminacy is obtained with a unique steady state when the elasticity of intertemporal substitution in consumption is large enough. Finally, we derive from Figure 7 that in case iii),  $\underline{\gamma}$  is equal to  $\gamma^F$  or  $\gamma^T$  depending on the value of  $\varepsilon_{rk}$ .

Notice also that in case i), the existence of local indeterminacy requires large enough values for the elasticity of the rental rate of capital, i.e. low

enough values for the sectoral elasticities of capital-labor substitution. On the contrary, in cases ii) and iii), the existence of multiple equilibrium paths is obtained without any restriction on these elasticities.

## 6 Extensions

Up to now we have assumed gross substitutability between the consumption levels  $c_t$  and  $d_{t+1}$ , *i.e.*,  $\gamma > 1$ . We consider in this Section that the saving function is decreasing with respect to the gross rate of return  $R$ :

**Assumption 4.**  $\gamma \in (0, 1)$

As previously, the fundamental properties of  $\Delta(\mathcal{T})$  are still characterized from the consideration of its extremities. The starting point of the pair  $(\mathcal{T}(\gamma), \mathcal{D}(\gamma))$  is now obtained when  $\gamma = 0$ :

$$\begin{aligned}\mathcal{D}(0) &= \mathcal{D}_0 = -\frac{s}{(1+n)b\alpha(1-s)} \\ \mathcal{T}(0) &= \mathcal{T}_0 = \frac{\varepsilon_{rk}[(1-s)\alpha - s(1+n)b] - (1-s)}{(1+n)b\varepsilon_{rk}\alpha(1-s)}\end{aligned}\quad (29)$$

while the end point is obtained when  $\gamma$  converges to 1 from below:

$$\begin{aligned}\lim_{\gamma \rightarrow 1^-} \mathcal{D}(\gamma) &= \mathcal{D}_1^- = \pm\infty \Leftrightarrow b[1 - \alpha + \alpha(1+n)b] \leq 0 \\ \lim_{\gamma \rightarrow 1^-} \mathcal{T}(\gamma) &= \mathcal{T}_1^- = \pm\infty \\ \Leftrightarrow b[(1-\alpha)(1-s) + \varepsilon_{rk}(1+n)bs[1 - \alpha + \alpha(1+n)b]] &\leq 0\end{aligned}\quad (30)$$

Moreover, we derive from (26) that  $\mathcal{D}'(\gamma) \geq 0$  if and only if  $\mathcal{D}_1^- = \pm\infty$ .

The next Lemma provides informations on the starting point  $(\mathcal{T}_\infty, \mathcal{D}_\infty)$  and  $\mathcal{D}'(\gamma)$ :

**Lemma 5.** *Under Assumptions 1, 2, 4, for given  $s$ ,  $\alpha$ ,  $b$  and  $\varepsilon_{rk}$ , the following results hold:*

- i)  $\mathcal{D}_0 < 0$  if and only if  $b > 0$ ;
- ii) When  $b > 0$ ,  $\mathcal{D}_0 \in (-1, 0)$  if and only if  $b > s/(1+n)(1-s)\alpha$ ;
- iii) When  $b < 0$ ,  $\mathcal{D}_0 \in (0, 1)$  if and only if  $b < -s/(1+n)(1-s)\alpha$ ;
- iv)  $\mathcal{D}'(\gamma) > 0$  if and only if  $b \in (-(1-\alpha)/(1+n)\alpha, 0)$ .

*Proof:* See Appendix 8.6. □

From these basic properties, our main objective is to obtain results that were not available under gross substitutability. More precisely, we want to derive conditions for local indeterminacy when the investment good is capital intensive and for the existence of a Hopf bifurcation when the consumption good is capital intensive. We are also interested in determining whether or not these configurations are compatible with dynamic efficiency of equilibria.

## 6.1 A capital intensive investment good

Let us start with the case  $b > 0$ . Lemma 5 shows that  $\mathcal{D}_0 < 0$  and  $\mathcal{D}'(\gamma) < 0$ . Therefore, denoting  $b_3 = s/(1+n)(1-s)\alpha$ , the existence of local indeterminacy fundamentally requires  $\mathcal{D}_0 \in (-1, 0)$ , *i.e.*,  $b > b_3$ . Notice that since the capital intensity difference necessarily satisfies  $b < 1/(1+n)$ ,<sup>12</sup> the inequality  $b > b_3$  implies an implicit restriction on the share of first period consumption over the wage income, namely  $\alpha > s/(1-s)$ . We have now to locate precisely the starting point  $(\mathcal{T}_0, \mathcal{D}_0)$  and the intersection of  $\Delta(\mathcal{T})$  with the ordinate axis.

**Lemma 6.** *Under Assumptions 1, 2, 4, let  $\underline{\alpha} = 1 - s/(1-s)$ ,  $b_3 = s/(1+n)(1-s)\alpha$  and  $b_4 = (1-\alpha)(1-s)/(1+n)s$ . For given  $s, \alpha, b$  and  $\varepsilon_{rk}$ , with  $\alpha > s/(1-s)$  and  $b > b_3$ , the following results hold:*

- i)  $1 + \mathcal{T}_0 + \mathcal{D}_0 < 0$  if and only if  $\varepsilon_{rk} < (1-s)/[1 + (1+n)b][\alpha(1-s) - s] \equiv \varepsilon_{rk}^1$ ;*
  - ii)  $1 - \mathcal{T}_0 + \mathcal{D}_0 > 0$  if and only if  $\varepsilon_{rk} < (1-s)/[1 - (1+n)b][\alpha(1-s) + s] \equiv \varepsilon_{rk}^2$ , with  $\varepsilon_{rk}^2 > \varepsilon_{rk}^1$ ;*
  - iii)  $\Delta(\mathcal{T}) = -1$  implies  $\mathcal{T} > 0$  if and only if  $\varepsilon_{rk} > (1+n)b[1 - \alpha(1-s)]/[1 - (1+n)^2b^2]s[1 - \alpha + \alpha(1+n)b] \equiv \varepsilon_{rk}^3$ , with  $\varepsilon_{rk}^3 > \varepsilon_{rk}^1$ .*
- Moreover,  $\varepsilon_{rk}^3 > \varepsilon_{rk}^2$  if and only if  $\alpha > \underline{\alpha}$  and  $b > \max\{b_3, b_4\}$ .*

*Proof:* See Appendix 8.7. □

Assume that  $\alpha > s/(1-s)$  and  $b > b_3$ . Depending on the location of  $(\mathcal{T}_0, \mathcal{D}_0)$ , local indeterminacy may a priori occur in three different configurations. Let us consider first the case  $\varepsilon_{rk} < \varepsilon_{rk}^1$  in which  $1 + \mathcal{T}_0 + \mathcal{D}_0 < 0$  and  $1 - \mathcal{T}_0 + \mathcal{D}_0 > 0$ . Since  $\mathcal{D}'(\gamma) < 0$ ,  $\Delta(\mathcal{T})$  will cross the interior of the triangle  $ABC$  if and only if  $\Delta(\mathcal{T}) = -1$  implies  $\mathcal{T} > 0$ , *i.e.*,  $\varepsilon_{rk} > \varepsilon_{rk}^3$ . But this is not possible since we know from Lemma 6 that  $\varepsilon_{rk}^3 > \varepsilon_{rk}^1$ .

Let us then consider the case  $\varepsilon_{rk} \in (\varepsilon_{rk}^1, \varepsilon_{rk}^2)$  in which  $1 + \mathcal{T}_0 + \mathcal{D}_0 > 0$  and  $1 - \mathcal{T}_0 + \mathcal{D}_0 > 0$ . Since  $\mathcal{D}_0 \in (-1, 0)$ ,  $(\mathcal{T}_0, \mathcal{D}_0)$  is then located within the triangle  $ABC$  and as  $\mathcal{D}'(\gamma) < 0$ , local indeterminacy occurs with low values for the elasticity of intertemporal substitution in consumption  $\gamma$ . A flip or a transcritical bifurcation will be obtained as  $\gamma$  is increased depending on whether  $\Delta(\mathcal{T}) = -1$  implies  $\mathcal{T} > 0$  or  $\mathcal{T} < 0$ . Notice also that, provided  $\alpha > s/(1-s)$ , the existence of sunspot fluctuations may be obtained with dynamic efficiency ( $\alpha > \underline{\alpha}$ ) or dynamic inefficiency  $\alpha \in (0, \underline{\alpha})$ . We get indeed the following Figure:

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<sup>12</sup>Denoting  $a \equiv k^0/l^0$ , linear homogeneity of  $T(k, y, \ell)$  implies  $a\ell/k = 1 - (1+n)b > 0$ .

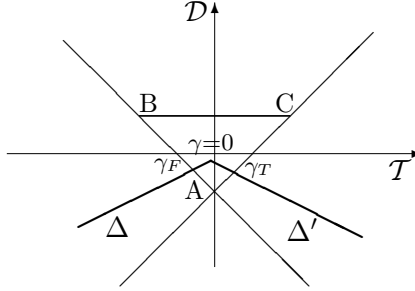


Figure 6:  $\alpha > s/(1-s)$ ,  $b > b_3$  and  $\varepsilon_{rk} \in (\varepsilon_{rk}^1, \varepsilon_{rk}^2)$ .

Let us finally consider the case  $\varepsilon_{rk} > \varepsilon_{rk}^2$  in which  $1 + \mathcal{T}_0 + \mathcal{D}_0 > 0$  and  $1 - \mathcal{T}_0 + \mathcal{D}_0 < 0$ . Since  $\mathcal{D}'(\gamma) < 0$ ,  $\Delta(\mathcal{T})$  will cross the interior of the triangle  $ABC$  if and only if  $\Delta(\mathcal{T}) = -1$  implies  $\mathcal{T} < 0$ , i.e.,  $\varepsilon_{rk} < \varepsilon_{rk}^3$ . As shown by Lemma 6, this last inequality is then compatible with  $\varepsilon_{rk} > \varepsilon_{rk}^2$  if and only if  $\alpha > \underline{\alpha}$  and  $b > \max\{b_3, b_4\}$ . We then get the following Figure:

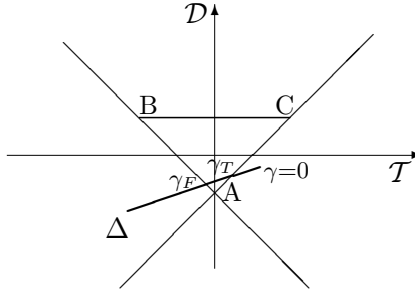


Figure 7:  $\alpha > \max\{s/(1-s), \underline{\alpha}\}$ ,  $b > \max\{b_3, b_4\}$  and  $\varepsilon_{rk} \in (\varepsilon_{rk}^2, \varepsilon_{rk}^3)$ .

Local indeterminacy here occurs with intermediary values for the elasticity of intertemporal substitution in consumption and is fundamentally related with multiple steady states and period-two cycles.

All these results can be summarized with the following Proposition:

**Proposition 6.** *Under Assumptions 1, 2, 4, let  $\underline{\alpha} = 1 - s/(1-s)$ ,  $b_3 = s/(1+n)(1-s)\alpha$  and  $b_4 = (1-\alpha)(1-s)/(1+n)s$ . There exist  $\underline{\varepsilon}_{rk} > 0$ ,  $\hat{\varepsilon}_{rk} > 0$ ,  $\bar{\varepsilon}_{rk} > 0$ ,  $\underline{\gamma} \in (0, 1)$  and  $\bar{\gamma} \in (0, 1)$  such that the NSS is locally indeterminate if and only if one of the following sets of conditions is satisfied:*

- i)  $\alpha > s/(1-s)$ ,  $b > b_3$ ,  $\varepsilon_{rk} \in (\underline{\varepsilon}_{rk}, \hat{\varepsilon}_{rk})$  and  $\gamma \in (0, \bar{\gamma})$ ,

ii)  $\alpha > \max\{s/(1-s), \underline{\alpha}\}$ ,  $b > \max\{b_3, b_4\}$ ,  $\varepsilon_{rk} \in (\hat{\varepsilon}_{rk}, \bar{\varepsilon}_{rk})$  and  $\gamma \in (\underline{\gamma}, \bar{\gamma})$ .

In both cases covered by Proposition 6, local indeterminacy can be obtained with dynamically efficient equilibria. However, intermediary values for the sectoral elasticities of capital-labor substitution are required.

## 6.2 A capital intensive consumption good

Let us now consider the case  $b > 0$ . Lemma 5 shows that  $\mathcal{D}_0 > 0$  and  $\mathcal{D}_0 \in (0, 1)$  if and only if  $b < -b_3$  with  $b_3 = s/(1+n)(1-s)\alpha$ . As shown by Figure 1, the existence of a Hopf bifurcation requires that  $\Delta(\mathcal{T})$  crosses the interior of segment  $BC$ . Depending on the sign of  $\mathcal{D}'(\gamma)$ , such an intersection can be obtained both with  $\mathcal{D}_0 \in (0, 1)$  or  $\mathcal{D}_0 > 1$ .

**Lemma 7.** *Under Assumptions 1, 2, 4, let  $\underline{\alpha} = 1 - s/(1-s)$ ,  $b_1 = -(1-\alpha)/(1+n)\alpha$  and  $b_3 = s/(1+n)(1-s)\alpha$ . For given  $s$ ,  $\alpha$ ,  $b$  and  $\varepsilon_{rk}$ , there exist  $\underline{\varepsilon}_{rk} > 0$  and  $\bar{\varepsilon}_{rk} > 0$  such that  $\Delta(\mathcal{T}) = 1$  implies  $\mathcal{T} \in (-2, 2)$  if and only if  $(\alpha - \underline{\alpha})(b - b_1) < 0$ ,  $(b - b_1)(b + b_3) < 0$  and  $\varepsilon_{rk} \in (\underline{\varepsilon}_{rk}, \bar{\varepsilon}_{rk})$ .*

*Proof:* See Appendix 8.8. □

Consider first the case with  $b < b_1$  and dynamic efficiency, *i.e.*,  $\alpha > \underline{\alpha}$ . We derive from Lemma 5 that  $b < b_1$  implies  $\mathcal{D}'(\gamma) < 0$ . Notice also that  $b_1 > -b_3$  if and only if  $\alpha > \underline{\alpha}$ . As a result, a Hopf bifurcation may only occur if  $\mathcal{D}_0 > 1$ , *i.e.*, if  $b \in (-b_3, b_1)$ . We get indeed the following geometrical representation with different  $\Delta$ -half-lines depending on the location of the starting point  $(\mathcal{T}_0, \mathcal{D}_0)$ :

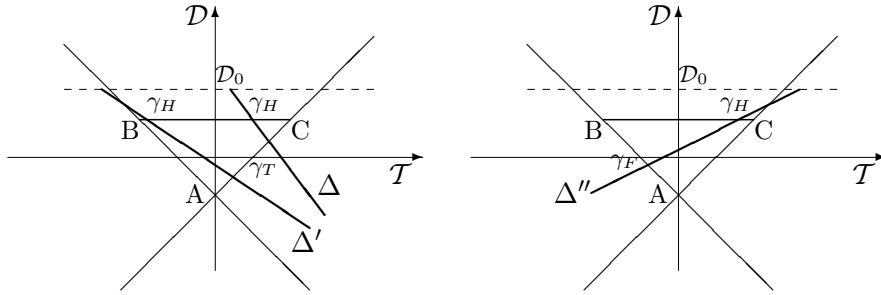


Figure 8:  $\alpha > \underline{\alpha}$ ,  $b \in (-b_3, b_1)$  and  $\varepsilon_{rk} \in (\underline{\varepsilon}_{rk}, \bar{\varepsilon}_{rk})$ .

Consider finally the case with  $b > b_1$  and dynamic inefficiency, *i.e.*,  $\alpha \in (0, \underline{\alpha})$ . It follows that  $b_1 < -b_3$ . Since  $\mathcal{D}'(\gamma) > 0$ , a Hopf bifurcation may only occur if  $\mathcal{D}_0 \in (0, 1)$ , *i.e.*, if  $b \in (b_1, -b_3)$ . We then get the following

geometrical representation with different  $\Delta$ -half-lines still depending on the location of the starting point  $(\mathcal{T}_0, \mathcal{D}_0)$ :

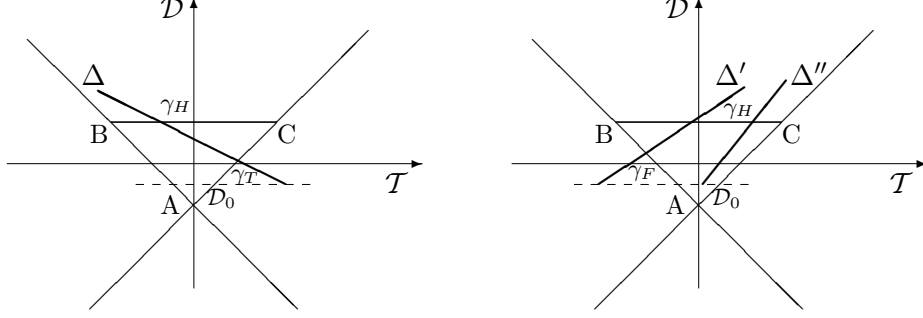


Figure 9:  $\alpha \in (0, \underline{\alpha})$ ,  $b \in (b_1, -b_3)$  and  $\varepsilon_{rk} \in (\underline{\varepsilon}_{rk}, \bar{\varepsilon}_{rk})$ .

All these results can be summarized with the following Proposition:

**Proposition 7.** *Under Assumptions 1, 2, 4, let  $\underline{\alpha} = 1-s/(1-s)$ ,  $b_1 = -(1-\alpha)/(1+n)\alpha$  and  $b_3 = s/(1+n)(1-s)\alpha$ . There exist  $\underline{\varepsilon}_{rk} > 0$ ,  $\bar{\varepsilon}_{rk} > 0$ ,  $\underline{\gamma} \in [0, 1)$ ,  $\gamma^H \in (0, 1)$  and  $\bar{\gamma} \in (0, 1)$  such that the NSS is locally indeterminate and  $\gamma^H$  is a Hopf bifurcation critical value if and only if one of the following sets of conditions is satisfied:*

- i)  $\alpha > \underline{\alpha}$ ,  $b \in (-b_3, b_1)$ ,  $\varepsilon_{rk} \in (\underline{\varepsilon}_{rk}, \bar{\varepsilon}_{rk})$  and  $\gamma \in (\gamma^H, \bar{\gamma})$ ,
- ii)  $\alpha \in (0, \underline{\alpha})$ ,  $b \in (b_1, -b_3)$ ,  $\varepsilon_{rk} \in (\underline{\varepsilon}_{rk}, \bar{\varepsilon}_{rk})$  and  $\gamma \in (\underline{\gamma}, \gamma^H)$ .

Notice that in case i),  $\bar{\gamma}$  is necessarily a flip or transcritical bifurcation value, while in case ii),  $\underline{\gamma}$  is equal to 0 if  $(\mathcal{T}_0, \mathcal{D}_0)$  is located within the triangle  $ABC$ .

As in Proposition 6, the existence of local indeterminacy requires intermediary values for the sectoral elasticities of capital-labor substitution. However, multiple equilibria are compatible with dynamic efficiency only in case i).

## 7 Related literature

The main comparisons have to be made with the papers of Galor [8] and Venditti [14] in which general formulations for the two-sector OLG model are considered. As shown by equation (16), any steady state within OLG models depends on preferences and technologies. As a result, as soon as an elasticity characterizing either the utility or the production functions is varied, the steady state itself is modified and this implies variations of all the other shares and elasticities. It follows that in Galor [8] and Venditti



[14], it is very difficult to know whether or not there exists a non-empty set of economies satisfying the conditions for the existence of local indeterminacy. In the current paper, based on the normalization procedure used to ensure the existence of a steady state, we may vary the elasticity of intertemporal substitution in consumption  $\gamma$  while keeping fixed the steady state and thus all the other shares and elasticities characterizing the preferences and technologies. Hence, all our results provide non-empty set of economies characterized by locally indeterminate equilibria.

Building on the main conclusions of Drugeon *et al.* [7], we also provide an integrated analysis of local indeterminacy and dynamic efficiency. Such a joint study is not provided in Galor [8] and Venditti [14] since dynamic efficiency is not discussed. We thus prove that there are many cases, either under gross substitutability or not, in which there exist a continuum of dynamically efficient equilibria.

Our main conclusions can be also compared with Ralf [11] and Reichlin [12] who consider two-sector OLG models with Leontief technologies. In Ralf [11], an example of local indeterminacy is provided under the assumptions of a capital intensive consumption good, a decreasing saving function with respect to the gross rate of return  $R$ , *i.e.*,  $\gamma \in (0, 1)$ , and under dynamic inefficiency since the stationary gross rate of return is less than 1. Considering our Proposition 7, this example shows that there exists some discontinuity between the case of regular technologies with positive elasticities of capital-labor substitution and the case with Leontief technologies. We have proved indeed that under dynamic inefficiency local indeterminacy requires some intermediary values for the sectoral elasticities of capital-labor substitution (see case ii) in Proposition 7).

Reichlin [12] does not provide an analysis of local indeterminacy *per se*. He gives however conditions for the existence of period-two cycles, through a flip bifurcation, and period-three cycles, giving rise to chaotic equilibrium paths. As a consequence, the existence of local indeterminacy can be deduced in a neighbourhood of the flip bifurcation. These conclusions are based on a capital intensive consumption good sector and a gross rate of return  $R$  lower than 1, *i.e.*, under dynamic inefficiency, and are compatible with the gross substitutability assumption. Our Proposition 5 then generalizes this result to the case of regular technologies with positive elasticities of capital-labor substitution.

## 8 Appendix

### 8.1 Proof of Proposition 1

Consider the set  $\mathcal{K}$  as defined by (14) and the expression of  $\alpha(R/B)$  as given by (12). Then  $\kappa^* \in (0, \bar{\kappa})$  is a solution of (16) if<sup>13</sup>

$$\frac{1}{1+\delta^\gamma \left( -\frac{T_1(\kappa^*, (1+n)\kappa^*, 1)}{T_2(\kappa^*, (1+n)\kappa^*, 1)B} \right)^{\gamma-1}} = 1 + (1+n) \frac{\kappa^* T_2(\kappa^*, (1+n)\kappa^*, 1)}{T_3(\kappa^*, (1+n)\kappa^*, 1)} \in (0, 1) \quad (31)$$

Let  $\delta = -T_2(\kappa^*, (1+n)\kappa^*, 1)B/T_1(\kappa^*, (1+n)\kappa^*, 1)$ . Equation (31) becomes

$$\frac{1}{1 - \frac{T_2(\kappa^*, (1+n)\kappa^*, 1)B}{T_1(\kappa^*, (1+n)\kappa^*, 1)}} = 1 + (1+n) \frac{\kappa^* T_2(\kappa^*, (1+n)\kappa^*, 1)}{T_3(\kappa^*, (1+n)\kappa^*, 1)} \quad (32)$$

so that the left-hand-side does not depend any more on  $\gamma$ . Moreover, there exists a unique value of  $B$  solution of (32) given by

$$B^* = \frac{(1+n)\kappa^* T_1(\kappa^*, (1+n)\kappa^*, 1)}{T_3(\kappa^*, (1+n)\kappa^*, 1) + (1+n)\kappa^* T_2(\kappa^*, (1+n)\kappa^*, 1)} > 0 \quad (33)$$

and  $\kappa^*$  is a steady state if and only if  $B = B^*$ .  $\square$

### 8.2 Proof of Lemma 1

From (12), one gets

$$\alpha'(R/B) = (1-\gamma)\alpha(R/B)(1-\alpha(R/B))\frac{B}{R} \quad (34)$$

It is shown in Benhabib and Nishimura [1, 2] and Bosi *et al.* [3] that

$$T_{12} = -T_{11}b, \quad T_{22} = T_{11}b^2 < 0, \quad T_{31} = -T_{11}a > 0, \quad T_{32} = T_{11}ab \quad (35)$$

with  $a \equiv k^0/l^0 > 0$  the capital-labor ratio in the consumption good sector,  $b$  the relative capital intensity difference across sectors as defined by (19) and  $T_{11} < 0$ . Let us then define the elasticity of the rental rate of capital

$$\varepsilon_{rk} = -T_{11}(\kappa^*, (1+n)\kappa^*, 1)\kappa^*/T_1(\kappa^*, (1+n)\kappa^*, 1)$$

the elasticity of the price of investment good

$$\varepsilon_{py} = T_{22}(\kappa^*, (1+n)\kappa^*, 1)(1+n)\kappa^*/T_2(\kappa^*, (1+n)\kappa^*, 1)$$

and the elasticity of the wage rate

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<sup>13</sup>From Definition 1, we derive indeed  $1 + (1+n)\kappa^* T_2(\kappa^*, (1+n)\kappa^*, 1)/T_3(\kappa^*, (1+n)\kappa^*, 1) = 1 - py/wl = 1 - \phi/w \in (0, 1)$ .

$$\varepsilon_{wk} = T_{31}(\kappa^*, (1+n)\kappa^*, 1)\kappa^*/T_3(\kappa^*, (1+n)\kappa^*, 1)$$

all evaluated at the NSS. Total differentiation of (15) using these expressions with (6), (7) and (34) evaluated at the NSS gives

$$\mathcal{D} = \frac{(1+n)b\varepsilon_{wk} + \varepsilon_{py}[1+\alpha(\gamma-1)]}{(1+n)^2b^2\varepsilon_{rk}\alpha(\gamma-1)}, \quad \mathcal{T} = \frac{1+(1+n)b\varepsilon_{wk} + \varepsilon_{py} + \alpha(\gamma-1)(\varepsilon_{rk} + \varepsilon_{py})}{(1+n)b\varepsilon_{rk}\alpha(\gamma-1)} \quad (36)$$

Considering (35) with  $T_1\kappa^*/T_3 = s/(1-s)$ ,  $-T_1/T_2 = R^* = s/(1-\alpha)(1-s)$  and the fact that the linear homogeneity of  $T(k, y, \ell)$  implies  $a = [1 - (1+n)b]\kappa^*$ , we derive

$$\varepsilon_{py} = \frac{\varepsilon_{rk}(1+n)^2b^2s}{(1-\alpha)(1-s)} \text{ and } \varepsilon_{wk} = \frac{\varepsilon_{rk}[1-(1+n)b]s}{(1-s)}$$

Substituting these expressions into (36) gives the result.  $\square$

### 8.3 Proof of Lemma 2

- i) We get from (18), (24) and Proposition 2 that  $\mathcal{D}_\infty = R^*/(1+n) > 1$  iff  $\alpha > \underline{\alpha}$ .
- ii) The result follows from (25) and (26).
- iii) The result immediately follows from (24) and Assumption 2.
- iv)-v) Obvious computations from (24) give

$$1 + \mathcal{T}_\infty + \mathcal{D}_\infty = [1 + (1+n)b] \frac{(1-\alpha)(1-s) + (1+n)bs}{(1+n)b(1-\alpha)(1-s)} \quad (37)$$

The result follows from the fact that  $-(1-\alpha)(1-s)/(1+n)s > -1$  if and only if  $\alpha > \underline{\alpha}$ .  $\square$

### 8.4 Proof of Lemma 3

Let  $b_0 = -1/(1+n)$ ,  $b_1 = -(1-\alpha)/(1+n)\alpha$  and  $b_2 = -(1-\alpha)(1-s)/(1+n)s$ .  
1 - Solving  $\mathcal{D} = 1$  in Lemma 1 gives

$$\alpha(1-\gamma) = \frac{\alpha s(b-b_1)}{b(1-s)(\alpha-\underline{\alpha})} \quad (38)$$

Under  $\alpha > \underline{\alpha}$ , since  $\gamma > 1$ , (38) can be satisfied if and only if  $b \in (b_1, 0)$ . Substituting  $\mathcal{D} = 1$  into the expression of  $\mathcal{T}$  allows to get

$$\mathcal{T} + 2 = \frac{1}{(1+n)b\varepsilon_{rk}\alpha(\gamma-1)} + \frac{[1+(1+n)b]^2}{(1+n)b} < 0 \quad (39)$$

2 - Solving  $\mathcal{D} = -1$  in Lemma 1 gives

$$\alpha(1-\gamma) = \frac{\alpha s(b-b_1)}{b[1-\alpha(1-s)]} \quad (40)$$

Since  $\gamma > 1$ , (44) can be satisfied if and only if  $b \in (b_1, 0)$ .

i) If  $\alpha > 1/2$  then  $b_1 > b_0$  and substituting  $\mathcal{D} = -1$  into the expression of  $\mathcal{T}$  allows to get under  $b \in (b_1, 0)$

$$\mathcal{T} = \frac{1}{(1+n)b\varepsilon_{rk}\alpha(\gamma-1)} + \frac{1-(1+n)^2b^2}{(1+n)b} < 0 \quad (41)$$

ii) and iii) Now let  $s \in (1/3, 1/2)$ , so that  $\underline{\alpha} < 1/2$ , and  $\alpha \in (\underline{\alpha}, 1/2)$ . It follows that  $b_1 < b_0$ . Substituting (44) into (41) gives

$$\mathcal{T} = \frac{\varepsilon_{rk}s(b-b_1)[1-(1+n)^2b^2]-b[1-\alpha(1-s)]}{(1+n)b\varepsilon_{rk}\alpha s(b-b_1)} \quad (42)$$

Therefore when  $b \in (b_0, 0)$ ,  $\mathcal{T} < 0$  but when  $b \in (b_1, b_0)$ ,  $\mathcal{T} < 0$  if

$$\varepsilon_{rk} < \frac{b(1-\alpha+\alpha s)}{[1-(1+n)^2b^2]\alpha s(b-b_1)} \equiv \underline{\varepsilon}_{rk} \quad (43)$$

3 - We finally derive from (42) that  $\mathcal{D} = -1$  implies  $\mathcal{T} > 0$  iff  $s \in (1/3, 1/2)$ ,  $\alpha \in (\underline{\alpha}, 1/2)$ ,  $b \in (b_1, b_0)$  and  $\varepsilon_{rk} > \underline{\varepsilon}_{rk}$ .  $\square$

## 8.5 Proof of Lemma 4

Let  $\alpha < \underline{\alpha}$ . We follow the same steps as in the proof of Lemma 3.

1 - Since  $\gamma > 1$ , we derive from (38) that  $\mathcal{D} = 1$  can be satisfied if and only if  $b \in (-\infty, b_1)$ . In such a case we derive from (39) that  $\mathcal{D} = 1$  implies  $\mathcal{T} < -2$ .

2 - Since  $\gamma > 1$ , we derive from (44) that  $\mathcal{D} = -1$  can be satisfied if and only if  $b \in (b_1, 0)$ . In such a case we derive from (41) and (42) that  $\mathcal{D} = -1$  implies  $\mathcal{T} < 0$  either when  $b > b_0$ , or when  $b < b_0$  and  $\varepsilon_{rk} \in (0, \underline{\varepsilon}_{rk})$ , with  $\underline{\varepsilon}_{rk}$  as defined by (43). Recall now that  $b_1 > b_0$  if and only if  $\alpha > 1/2$ , and  $\underline{\alpha} > 1/2$  if and only if  $s \in (0, 1/3)$ . We derive from all this that  $\mathcal{D} = -1$  implies  $\mathcal{T} < 0$  in the following cases:

- \* when  $s \in (0, 1/3)$ ,  $\alpha \in (0, 1/2)$  and  $b \in (b_0, 0)$ ,
- \* when  $s \in (0, 1/3)$ ,  $\alpha \in (0, 1/2)$ ,  $b \in (b_1, b_0)$  and  $\varepsilon_{rk} \in (0, \underline{\varepsilon}_{rk})$ ,
- \* when  $s \in (0, 1/3)$ ,  $\alpha \in (1/2, \underline{\alpha})$ ,  $b \in (b_1, 0)$ ,
- \* when  $s \in (1/3, 1/2)$ ,  $\alpha \in (0, \underline{\alpha})$ ,  $b \in (b_0, 0)$ ,
- \* when  $s \in (1/3, 1/2)$ ,  $\alpha \in (0, \underline{\alpha})$ ,  $b \in (b_1, b_0)$  and  $\varepsilon_{rk} \in (0, \underline{\varepsilon}_{rk})$ .

The result follows from summarizing all these subcases.

3 - We finally derive from all the previous subcases that  $\mathcal{D} = -1$  implies  $\mathcal{T} > 0$  iff  $\alpha \in (0, \min\{\underline{\alpha}, 1/2\})$ ,  $b \in (b_1, b_0)$  and  $\varepsilon_{rk} > \underline{\varepsilon}_{rk}$ .  $\square$

## 8.6 Proof of Lemma 5

i)-iv) The results immediately follow from (29) and (26).

## 8.7 Proof of Lemma 6

From (29) we derive

$$\begin{aligned} 1 + \mathcal{T}(0) + \mathcal{D}_0 &= \frac{\varepsilon_{rk}[1+(1+n)b][(1-s)\alpha-s]-(1-s)}{(1+n)b\varepsilon_{rk}\alpha(1-s)} \\ 1 - \mathcal{T}(0) + \mathcal{D}_0 &= \frac{1-s-\varepsilon_{rk}[1-(1+n)b][(1-s)\alpha+s]}{(1+n)b\varepsilon_{rk}\alpha(1-s)} \end{aligned}$$

i) and ii) immediately follows from  $\alpha > s/(1-s)$  and  $b > b_3$ . Moreover, we have

$$\varepsilon_{rk}^1 - \varepsilon_{rk}^2 = \frac{2(1+n)\alpha(1-s)^2(b_3-b)}{[1-(1+n)^2b^2][(1-s)^2\alpha^2-s^2]} < 0$$

iii) Solving  $\mathcal{D} = -1$  in Lemma 1 gives

$$\alpha(1-\gamma) = \frac{s[1-\alpha+\alpha(1+n)b]}{(1+n)b[1-\alpha(1-s)]} \quad (44)$$

Substituting this into the expression of  $\mathcal{T}$  allows to get

$$\mathcal{T} = \frac{-[1-\alpha(1-s)]}{\varepsilon_{rk}s[1-\alpha+\alpha(1+n)b]} + \frac{1-(1+n)^2b^2}{(1+n)b}$$

It follows that  $\mathcal{T} > 0$  if and only if  $\varepsilon_{rk} > \varepsilon_{rk}^1$ . Moreover,  $\varepsilon_{rk}^3 > \varepsilon_{rk}^1$  is equivalent to

$$\alpha s(1-s)(1+n)^2b^2 + (1+n)b[\alpha(1-\alpha)(1-s)^2 - s^2] - (1-s)s(1-\alpha) > 0$$

This inequality is ensured by the condition  $b > b_3$ . Similarly,  $\varepsilon_{rk}^3 > \varepsilon_{rk}^2$  is equivalent to

$$\alpha s(1-s)(1+n)^2b^2 - (1+n)b[\alpha(1-\alpha)(1-s)^2 + s^2] + (1-s)s(1-\alpha) > 0$$

The two roots of this polynomial are equal to  $b_3$  and  $b_4$ . Notice then that  $b_4 < 1/(1+n)$  if and only if  $\alpha > \underline{\alpha}$ . It follows therefore that the polynomial is positive if  $\alpha > \underline{\alpha}$  and  $b > \max\{b_3, b_4\}$ .  $\square$

## 8.8 Proof of Lemma 7

Solving  $\mathcal{D} = 1$  in Lemma 1 gives

$$\alpha(1-\gamma) = \frac{\alpha s(b-b_1)}{b(1-s)(\alpha-\underline{\alpha})}$$

Substituting this into the expression of  $\mathcal{T}$  allows to get

$$\begin{aligned} \mathcal{T} - 2 &= \frac{(1+n)(b+b_1)^2}{b} - \frac{(1-s)(\alpha-\underline{\alpha})}{(1+n)\varepsilon_{rk}\alpha s(b-b_1)} \\ \mathcal{T} + 2 &= \frac{(1+n)(b-b_1)^2}{b} - \frac{(1-s)(\alpha-\underline{\alpha})}{(1+n)\varepsilon_{rk}\alpha s(b-b_1)} \end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{T} < 2 &\Leftrightarrow \varepsilon_{rk} > \frac{(1-s)(\alpha-\underline{\alpha})b}{(1+n)^2(b+b_1)^2\alpha s(b-b_1)} \equiv \underline{\varepsilon}_{rk} \\ \mathcal{T} > -2 &\Leftrightarrow \varepsilon_{rk} < \frac{(1-s)(\alpha-\underline{\alpha})b}{(1+n)^2(b-b_1)^2\alpha s(b-b_1)} \equiv \bar{\varepsilon}_{rk}\end{aligned}$$

We then get

$$\bar{\varepsilon}_{rk} - \underline{\varepsilon}_{rk} = \frac{(1-s)(\alpha-\underline{\alpha})4b^2b_1}{(1+n)^2(b-b_1)^2(b+b_1)^2\alpha s(b-b_1)}$$

Since  $b_1 < 0$ ,  $\bar{\varepsilon}_{rk} - \underline{\varepsilon}_{rk} > 0$  if and only if  $(\alpha - \underline{\alpha})(b - b_1) < 0$ . Recall now that when  $b < 0$ ,  $\mathcal{D}_0 > 0$  and  $\mathcal{D}_0 \in (0, 1)$  if and only if  $b < -b_3$ . Moreover,  $\mathcal{D}'(\gamma) > 0$  if and only if  $b \in (b_1, 0)$ . Therefore,  $\Delta(\mathcal{T})$  will cross the interior of segment  $BC$  only in the following two cases: i) if  $\mathcal{D}_0 > 1$  and  $\mathcal{D}'(\gamma) < 0$ , *i.e.*,  $b > -b_3$  and  $b < b_1$ , or ii) if  $\mathcal{D}_0 \in (0, 1)$  and  $\mathcal{D}'(\gamma) > 0$ , *i.e.*,  $b < -b_3$  and  $b > b_1$ . These two configurations may be summarized by the condition  $(b + b_3)(b - b_1) < 0$ . The proof is finally completed by noting that  $b_1 > -b_3$  if and only if  $\alpha > \underline{\alpha}$ .  $\square$

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